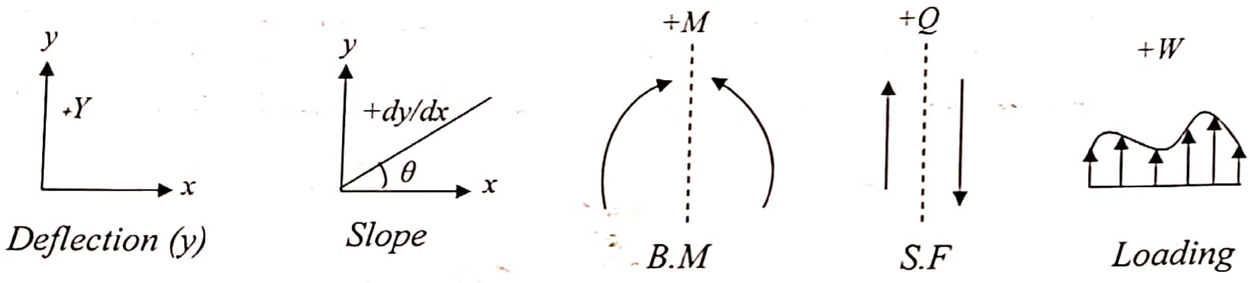
# Slope and deflection of beams

Convention

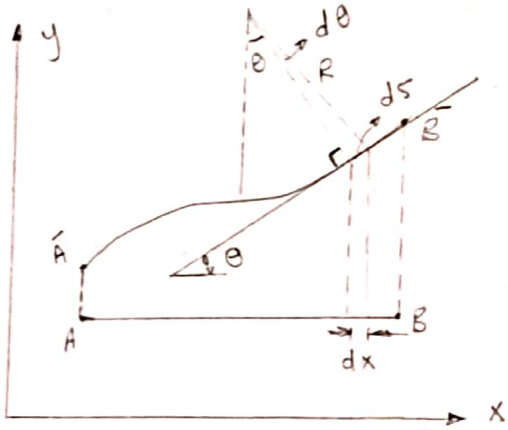



Relations between (loading ,shear force ,bending moment ,slope and deflection)

$\theta = \frac{d_y}{d_x} \equiv slope$ Since: $\theta$ is small thus $d_s = d_x$ but $d_s = Rd\theta \Rightarrow Rd\theta = dx \Rightarrow \frac{d\theta}{dx} = \frac{1}{R}$

$\therefore \frac{d^2y}{dx^2} = \frac{1}{R} = \frac{M}{EI} \Rightarrow M = EI\frac{d^2y}{dx^2}$ and $Q = \frac{dM}{dx} \Rightarrow Q = EI\frac{d^3y}{dx^3}$

Finally: $Q = \int w dx \Rightarrow w = \frac{dQ}{dx} \Rightarrow W = EI\left(\frac{d^4y}{dx^4}\right)$




Macaulays method (direct integration)

$M = EI\frac{d^2y}{dx^2} \Rightarrow \frac{d^2y}{dx^2} = \frac{1}{EI} \cdot M$ Integrating

$\frac{dy}{dx} = \frac{1}{EI}\int M_{dx} + A$

$Y = \frac{1}{EI}\int M_{dx} + Ax + B$ Where A and B constants to be found from boundary conditions

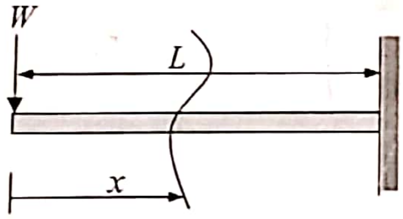
## Type of loading:

(1) Cantilever with concentrated load:

$EIy'' = M = -wx$

$EIy' = -\frac{1}{2}wx^2 + A \quad \rightarrow (1)$

$EIy = -\frac{1}{6}wx^3 + Ax + B \quad \rightarrow (2)$




Boundary conditions

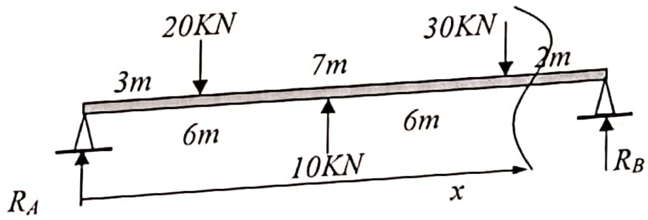
2. $At\ x=0 \Rightarrow M=0 \Rightarrow B=0$

3. $At\ x=L \Rightarrow y'=0 \Rightarrow C=\frac{wL^3}{4}$

4. $At\ x=L \Rightarrow y=0 \Rightarrow D=\frac{23}{120}wL^4$

$$y=\frac{1}{EI}\left(-\frac{wx^4}{24}-\frac{wx^5}{60L}+\frac{wL^3x}{4}-\frac{23}{120}wL^4\right)$$

➤ <u>*Simply supported beam with concentrated loads:*</u>




$$\sum M_B=0 \Rightarrow R_A=15KN$$
$$\sum Fy=0 \Rightarrow R_B=25KN$$

$$EIy''=M=15x-20(x-3)+10(x-6)-30(x-10)$$

$$EIy'=\frac{15}{2}x^2-10(x-3)^2+5(x-6)^2-15(x-10)^2+A \quad \rightarrow(1)$$

$$EIy=\frac{5}{2}x^3-\frac{10}{3}(x-3)^3+\frac{5}{3}(x-6)^2-5(x-10)^3+Ax+B \quad \rightarrow(2)$$

*Boundary conditions*

1. $At x=0 \Rightarrow y=0 \Rightarrow B=0$

*Note:* (x-3) = 0 for $x \le 3$
(x-6) = 0 for $x \le 6$
(x-10) = 0 for $x \le 10$

2. $At x=12 \Rightarrow y=0 \Rightarrow 0=\frac{5}{2}(12)^3-\frac{10}{3}(9)^3\frac{5}{3}(6)^3-5(2)^3+12A \Rightarrow A=-184.2$

$$EIy=\frac{5}{2}x^3-\frac{10}{3}(x-3)^3+\frac{5}{3}(x-6)^3-5(x-10)^3-184.2x$$

*Note : Unit used are ( KN,m)*

*The deflection at the center x=6m* $\Rightarrow (x-6)=0$
$(x-10)=0$

$$EIy=\frac{5}{2}(6)^3-\frac{10}{3}(3)^3-184.2(6)=-655.2KN.m^3$$

*For E= 208 GPa and* $I=82*10^{-6}m^4$

$Y=38.4*10^{-3}m=38.4\ mm$

1. $At x = L \Rightarrow y' = 0 \Rightarrow 0 = \frac{-1}{2}wL^2 + A \Rightarrow A = \frac{1}{2}wL^2$
2. $At x = L \Rightarrow y = 0 \Rightarrow 0 = \frac{-1}{6}wL^3 + \frac{1}{2}wL^2 \cdot L + B \Rightarrow B = \frac{-1}{3}wL^3$

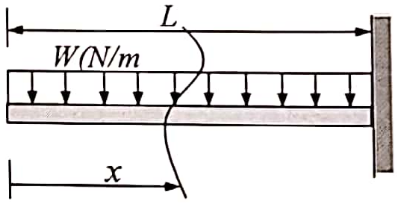
$\therefore EIy = -\frac{1}{6}wx^3 + \frac{1}{2}wL^2x - \frac{1}{3}wL^3$

$At x = 0 \Rightarrow y = y_{max} \Rightarrow EIy_{max} = -\frac{1}{3}wL^3 \Rightarrow y_{max} = -\frac{wL^3}{3EI}$

## cantilever beam with uniformly distributed load (U.D.L):

$EIy'' = M = -wx.\frac{x}{2}$

$EIy' = -\frac{1}{6}wx^3 + A$




$EIy = -\frac{1}{24}wx^4 + Ax + B$

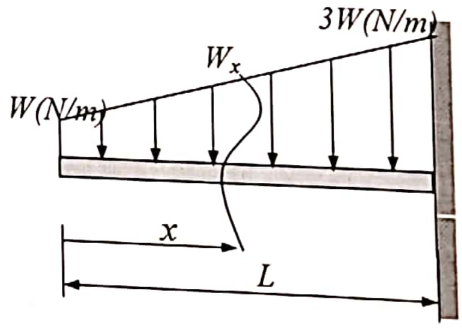
Boundary conditions

1. $At x = L \Rightarrow y' = 0 \Rightarrow -\frac{1}{6}wL^3 + A \Rightarrow A = \frac{1}{6}wL^3$
2. $At x = L \Rightarrow y = 0 \Rightarrow -\frac{1}{24}wL^4 + \frac{1}{6}wL^3.L + B \Rightarrow B = -\frac{1}{8}wL^4$

$\therefore y = \frac{1}{EI}\left(-\frac{1}{24}wx^4 + \frac{1}{6}wL^3x - \frac{1}{8}wL^4\right)$

$At x = 0 \Rightarrow y = y_{max} = -\frac{wL^4}{8EI} \qquad y'_{max} = \frac{wL^3}{6EI}$

## cantilever beam with increasing (U.D.L):




$wx = EIy'''' = -\left(w + \frac{2w}{L}x\right)$

$EIy''' = -wx - \frac{w}{L}x^2 + A = Q \qquad \rightarrow (1)$

$EIy'' = -\frac{1}{2}wx^2 - \frac{w}{3L}x^3 + Ax + B = M \qquad \rightarrow (2)$

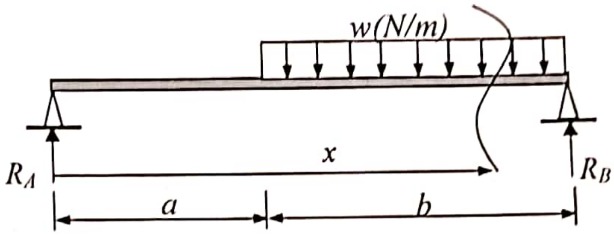
$EIy' = -\frac{1}{6}wx^3 - \frac{w}{12L}x^4 + \frac{1}{2}Ax^2 + Bx + C = \theta \qquad \rightarrow (3)$

$EIy = -\frac{1}{24}wx^4 - \frac{w}{60L}x^5 + \frac{1}{6}Ax^3 + \frac{1}{2}Bx^2 + Cx + D \qquad \rightarrow (4)$

Boundary conditions

1. $At x=0 \Rightarrow Q = 0 \Rightarrow A = 0$

## Simply supported beam with (U.D.L):




$\sum M_B = 0 \quad \sum Fy = 0$

$R_A$ and $R_B$ are found

$\therefore EIy'' = M = R_A.x - w(x-a).\frac{(x-a)}{2}$

$= R_A.x - \frac{1}{2}w(x-a)^2$

$EIy' = \frac{1}{2}R_A x^2 - \frac{1}{6}w(x-a)^3 + A$

$EIy = \frac{1}{6}R_A x^3 - \frac{1}{24}w(x-a)^4 + Ax + B$

Boundary conditions:

1. At $x = 0 \Rightarrow y = 0 \Rightarrow B = 0$
2. At $x = a + b = L \Rightarrow y = 0 \Rightarrow B = 0 = \frac{1}{6}RaL^3 - \frac{1}{24}w(x-a)^4 + AL \Rightarrow A =$

## Simply supported with (U.D.L) over a part of the beam:

$EIy'' = M = RA.x - w(x-a).\frac{(x-a)}{2} + w(x-b).\frac{(x-b)}{2}$

$= R_A.x - \frac{1}{2}w(x-a)^2 + \frac{1}{2}w(x-b)^2$

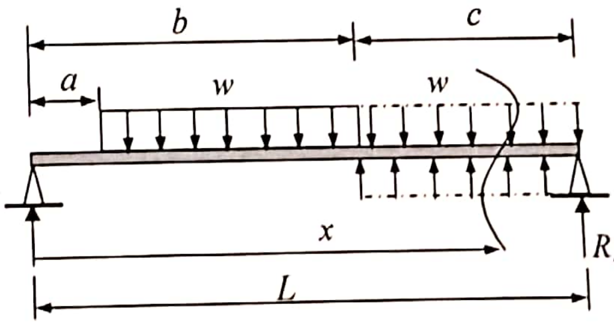
$EIy' = \frac{1}{2}R_A x^2 - \frac{1}{6}w(x-a)^3 + \frac{1}{6}w(x-b)^3 + A$

$EIy = \frac{1}{6}R_A x^3 - \frac{1}{24}w(x-a)^4 + \frac{1}{24}w(x-b)^4 + Ax + B$

Boundary conditions:

1. At $x = o \Rightarrow y = 0$
2. At $x = L \Rightarrow y = 0$

From these equation A and B are found

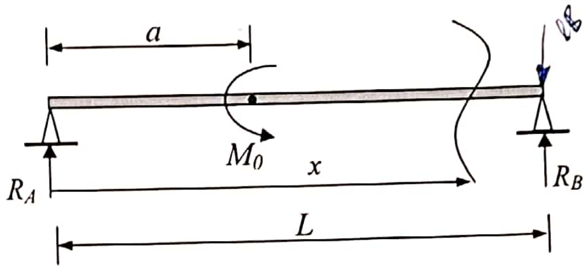


## Simply supported beam with a couple :

$\sum M_B = 0$

$\sum Fy = 0$

$R_A$ and $R_B$ are found




$$EIy'' = M = R_A.x - M_0(x-a)^0$$

$$EIy' = \frac{1}{2}R_A x^2 - \frac{1}{2}M_0(x-a) + A$$

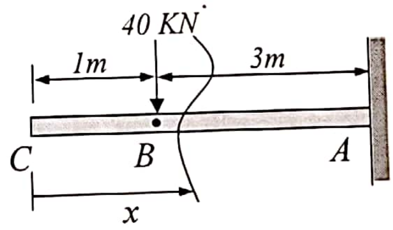
$$EIy = \frac{1}{6}R_A x^3 - \frac{1}{2}M_o(x-a)^2 + Ax + B$$

Boundary conditions :

1. $At\ x = 0 \Rightarrow y = 0 \Rightarrow B = 0$

2. $At\ x = L \Rightarrow y = 0 \Rightarrow 0 = \frac{1}{6}R_A L^3 - \frac{1}{2}M_o(L-a) + AL \Rightarrow A =$

Example (1): Find $y_c$ for the beam shown in the figure?




EI= 65 $MN/m^2$

Solution:

$$EIy'' = M = -40(x-1)$$

$$EIy' = -20(x-1)^2 + A \rightarrow (1)$$

$$EIy = -\frac{20}{3}(x-1)^3 + Ax + B \rightarrow (2)$$

Boundary conditions:

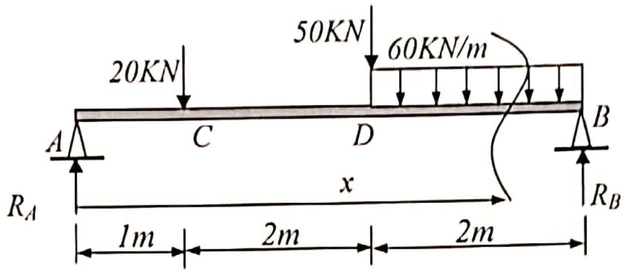
1. $At\ x = 4 \Rightarrow y' = 0 = -20(3)^2 + A$

$\Rightarrow A = 180$

2. $At\ x = 4 \Rightarrow y = 0 \Rightarrow 0 = -\frac{20}{3}(3)^3 + 180(4) + B \Rightarrow B = -540$

$$At\ x = 0 \Rightarrow y_C = \frac{1}{EI}\left(-0 + 0 - 540 = \frac{-540}{65 * 10^3}\right) = -8.31 * 10^{-3}\ m$$

Example: (2) find $y_d$ and $y_{max}$ for the beam shown in the figure if $E= 200GN/m^2$ and $I=83*10^{-6}\ m^4$ ?




Solution:

$$\sum M = o \Rightarrow RA = 60KN$$

$$\sum Fy = 0 \Rightarrow RB = 130KN$$

$$EIy'' = M = 60x - 20(x-1) - 50(x-3) - \frac{1}{2}.60(x-3)^2$$

$$EIy' = M = 30x^2 - 10(x-1)^2 - 25(x-3)^2 - 10(x-3)^3 + A$$

$$EIy = M = 10x^3 - \frac{10}{3}(x-1)^3 - \frac{25}{3}(x-3)^3 - \frac{10}{4}(x-3)^4 + Ax + B$$

Boundary conditions:

1. $At\ x = 5 \Rightarrow y = 0 \Rightarrow 0 = 10(50)^3 - \frac{10}{3}(4)^3 - \frac{25}{3}(2)^4 + 5A \Rightarrow A = -186$

2. $At\ x = 3 \Rightarrow EIy_d = 10(3)^3 - \frac{10}{3}(2)^3 - 186*3 \Rightarrow y_d = -19mm$

② at $x = 0$ $y = 0$ ⇒ $B = 0$

$$EIy'_D = 30(3)^2 - 10(2)^2 - 186 = 44$$

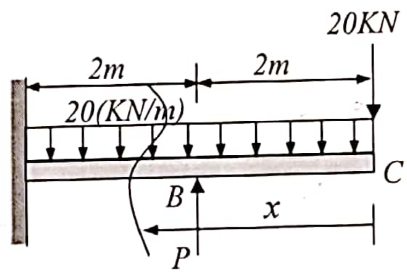
$$EIy'_C = 30(1)^2 - 186 = -156$$

y max occurs between C and D $(1 \succ x \succ 3)$

$$EIy' = 30x^2 - 10(x-1)^2 - 186 = 0 \Rightarrow x = 2.67m$$

$$\therefore EIy_{max} = 321.8KN.m^3 \Rightarrow y_{max} = \frac{321.8}{200*10^6*83*10^{-6}} = -19.4mm$$

Example (3): Find $y_C$ and $y'_C$ if $p=0$ and $EI= 20\ MN/m^2$




Solution:

$$EIy'' = M = -20x - 20x.\frac{x}{2} = -20x - 10x^2$$

$$EIy' = -10x^2 - \frac{10}{3}x^3 + A$$

$$EIy = -\frac{10}{3}x^3 - \frac{10}{12}x^4 + Ax + B$$

Boundary conditions : (1) $At\ x = 4m \Rightarrow y' = 0 = -10(4)^2 - \frac{10}{3}(4)^3 + A \Rightarrow A = 373.3$

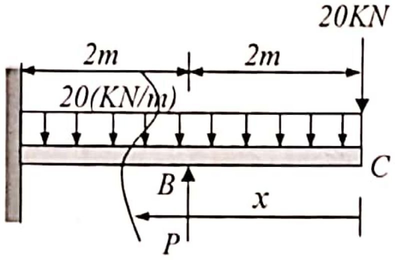
(2) $At\ x = 4m$

$$\Rightarrow y = 0 = -\frac{10}{3}(4)^3 - \frac{5}{6}(4)^4 + 4*373.3 \Rightarrow B = -1066.6$$

$$y'_c atx = 0 = \frac{1}{20*10^3}(373.3) = 18.667*10^{-3} rad$$

$$y_c atx = 0 = \frac{-1}{20*10^3}(1066.6) = -53*10^{-3} m$$

Example: (3B): Find the value of the force P to reduce the deflection of point c to the half?




Solution :

$$EIy'' = M = -20x - 20x.\frac{x}{2} = -20x - 10x^2 + P(x-2)$$

$$EIy' = -10x^2 - \frac{10}{3}x^3 + \frac{1}{2}P(x-2)^2 + A$$

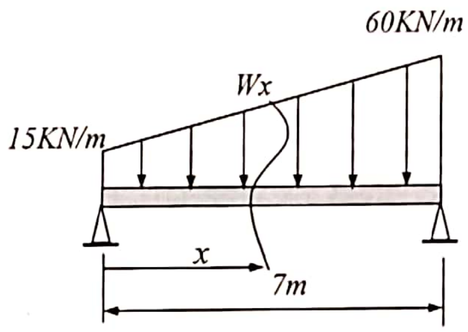
$$EIy = -\frac{10}{3}x^3 - \frac{10}{12}x^4 + \frac{1}{6}P(x-2)^3 + Ax + B$$

Boundary conditions:

1. $At\ x = 0 \Rightarrow y = 26.5*10^{-3} m \Rightarrow B = -20*10^3*26.5*10^{-3} = -530$
2. $At\ x = 4\ m \Rightarrow y = 0 = -\frac{10}{3}(4)^3 - \frac{5}{6}(4)^4 + \frac{1}{6}P(2)^3 + 4A - 530 \rightarrow (1)$
3. $At\ x = 4\ m \Rightarrow y' = 0 = -10(4)^2 - \frac{10}{3}(4)^3 + \frac{1}{2}P(2)^2 + A \rightarrow (2)$

From (1) and (2) P= 80 KN

Example (4): Find y at the mid-point if d= 450 mm $\sigma_{max}$ = 100 $MN/m^2$ and E= 210 $GN/m^2$




Solution : $EIy'''' = Wx = -15 - \frac{45x}{L}$

$$EIy''' = -15x - \frac{45x^2}{2L} + A = Qx$$

$$EIy'' = -\frac{15}{2}x^2 - \frac{45}{6L}x^3 + Ax + B = Mx$$

$$EIy' = -\frac{15}{6}x^3 - \frac{45}{24L}x^4 + \frac{1}{2}Ax^2 + Bx + C$$

$$EIy = -\frac{15}{24}x^4 - \frac{45}{120L}x^5 + \frac{1}{6}Ax^3 + \frac{1}{2}Bx^2 + Cx + D$$

Boundary conditions:

1. $At\ x = 0 \Rightarrow y = 0 \Rightarrow D = 0$

2. $At\ x = 0 \Rightarrow M = 0 \Rightarrow B = 0$

3. $At\ x = 7m \Rightarrow M = 0 = -7.5(7)^2 - \frac{45}{7*6}(7)^3 + 7A \Rightarrow A = 105$

4. $At\ x = 7m \Rightarrow y = 0 = -\frac{15}{24}(7)^4 - \frac{45}{120*7}(7)^5 + \frac{1}{6}(105)(7)^3 + 7C \Rightarrow C = -514.5$

$$\therefore EIy = -\frac{15}{24}x^4 - \frac{45}{120L}x^5 + \frac{105}{6}x^3 - 514.5x$$

$$EIy'' = -15x - \frac{45}{2L}x^2 + 105$$

$At\ x = 3.5\ m \quad EI\ y = -1172.35$

a) $\sigma_{max} = \frac{M_{max} \cdot y_{max}}{I}$

$M_{max}$ Occur at $Q_x = 0 \Rightarrow -15x - \frac{45}{2L}x^2 + 105 = 0$

$\Rightarrow x^2 + 4.667x - 32.667 = 0 \Rightarrow x_1 = -8.5m(Neglect)$

$x_2 = 3.8m$

Substituting this value in equation of $M_x$

$$M_{max} = -\frac{15}{2}(3.84)^2 - \frac{45}{6*7}(3.84)^3 + 105(3.84) = 232\,KN.m$$

$$100*10^6 = \frac{232*10^3*\frac{450}{2}*10^{-3}}{I} \Rightarrow I = 552*10^{-6}\,m^4$$

b) $552*10^{-6}*210*10^6\,y = -1172.36 \Rightarrow y = -10.67\,mm$

at x = 3.5 m mid point

# INDETERMINATE BEAMS

**DEFINITIONS:** *They are beams with extra supports. The reactions at these supports can not be determined using the equations of equilibrium only, rather the deflection and slope of beams must be concerned.*

1. *Built – in beams with concentrated loads :*

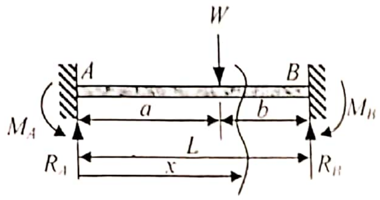
$$\sum Fy=0 \Rightarrow R_A+R_B=W \rightarrow (1)$$

$$\sum M_B=0 \Rightarrow -M_A+R_A.L-W.b+M_B=0 \rightarrow (2)$$

$$EIy''=M_x=-M_A x^0+R_A x-W(x-a)$$

$$EIy'=-M_A x+\frac{1}{2}R_A x^2-\frac{1}{2}W(x-a)^2+A$$

$$EIy=-\frac{1}{2}M_A x^2+\frac{1}{6}R_A x^3-\frac{1}{6}W(x-a)^3+Ax+B$$




*Boundary conditions:*

1. At $x=0 \Rightarrow y=0 \Rightarrow B=0$
2. At $x=0 \Rightarrow y'=0 \Rightarrow A=0$
3. At $x=L \Rightarrow y'=0=-M_A L+\frac{1}{2}R_A L^2-\frac{1}{2}W(L-a)^2 \rightarrow (3)$
4. At $x=L \Rightarrow y=0=-\frac{1}{2}M_A L^2+\frac{1}{6}R_A L^3-\frac{1}{6}W(L-a)^3 \rightarrow (4)$

*Four equations with Four unknown ( $R_A, R_B, M_A, M_B$)*

2. *Simply supported beam with concentrated loads:*

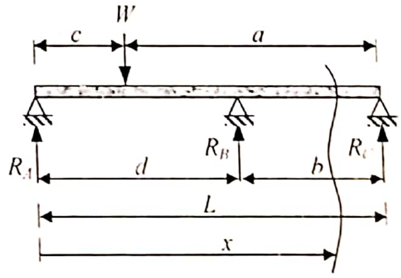
$$\sum Fy=0 \Rightarrow R_A+R_B+R_C=W \rightarrow (1)$$

$$\sum M_C=0 \Rightarrow R_A.L-W.a+R_B.b=0 \rightarrow (2)$$

$$EIy''=M=R_A x-W(x-c)+R_B(x-d)$$

$$EIy'=\frac{1}{2}R_A x^2-\frac{1}{2}W(x-c)^2+\frac{1}{2}R_B(x-d)^2+A$$

$$EIy=\frac{1}{6}R_A x^3-\frac{1}{6}W(x-c)^3+\frac{1}{6}R_B(x-d)^3+Ax+B$$

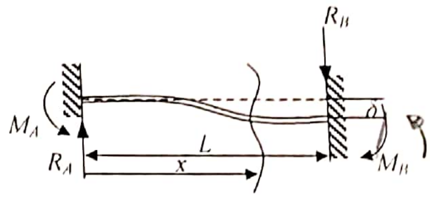



*Boundary conditions:*

1. At $x=0 \Rightarrow y=0 \Rightarrow B=0$
2. At $x=d \Rightarrow y=0=\frac{1}{6}R_A d^3-\frac{1}{6}W(d-c)^3+Ad \rightarrow (3)$
3. At $x=L \Rightarrow y=0=\frac{1}{6}R_A L^3-\frac{1}{6}W(L-c)^3+\frac{1}{6}R_B(L-d)^3+AL \rightarrow (4)$

*Four equations with Four unknown ( $R_A$, $R_B$, $R_C$, A)*

3. Built – in beams with movement of support:




$$\sum Fy = 0 \quad \Rightarrow R_A = R_B \quad \rightarrow (1)$$

$$\sum MB = 0 \quad \Rightarrow M_A + M_B = R_A.L \quad \rightarrow (2)$$

$$EIy'' = M = -M_A x^0 + R_A x$$

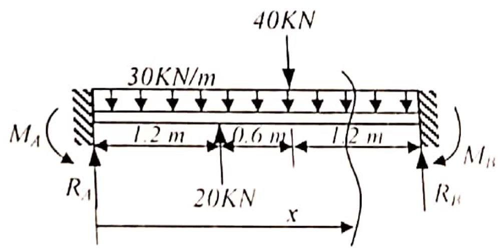
$$EIy' = -M_A x + \frac{1}{2} R_A x^2 + A$$

$$EIy = -\frac{1}{2} M_A x^2 + \frac{1}{6} R_A x^3 + Ax + B$$

Boundary conditions:

1. At $x = 0 \quad \Rightarrow y = 0 \quad \Rightarrow B = 0$
2. At $x = 0 \quad \Rightarrow y' = 0 \quad \Rightarrow A = 0$
3. At $x = L \quad \Rightarrow y = \delta \Rightarrow EI\delta = -\frac{1}{2} M_A L^2 + \frac{1}{6} R_A L^3 \quad \rightarrow (3)$
4. At $x = L \quad \Rightarrow y' = 0 \Rightarrow 0 = -M_A L + \frac{1}{2} R_A L^2 \quad \rightarrow (4)$

Four equations with Four unknown ( $R_A, R_B, M_A, M_B$)




Example (1): $I = 42 * 10^{-6}\ m^4 \qquad y_{max} = 100\,mm$

Find $\sigma_{max}$.

Solution:

$$\sum Fy = 0 \quad \Rightarrow R_A + R_B + 20 = 40 + 30 * 3 \rightarrow (1)$$

$$\sum MB = 0 \quad \Rightarrow RA * 3 - MA + 20 * 1.8 - 40 * 1.2 - 30 * 3 * 1.5 + MB = 0 \quad \rightarrow (2)$$

$$EIy'' = M = -M_A x^0 + R_A x + 20(x - 1.2) - 40(x - 1.8) - 30x * \frac{x}{2}$$

$$EIy' = -M_A x + \frac{1}{2} R_A x^2 + 10(x - 1.2)^2 - 20(x - 1.8)^2 - 5x^3 + A$$

$$EIy = -\frac{1}{2} M_A x^2 + \frac{1}{6} R_A x^3 + \frac{10}{3}(x - 1.2)^3 - \frac{20}{3}(x - 1.8)^3 - \frac{5}{4} x^4 + Ax + B$$

Boundary conditions:

1. At $x = 0 \quad \Rightarrow y = 0 \quad \Rightarrow B = 0$
2. At $x = 0 \quad \Rightarrow y' = 0 \quad \Rightarrow A = 0$
3. At $x = 3m \quad \Rightarrow y' = 0 = -M_A(3) + \frac{1}{2} R_A(3)^2 + 10(1.8)^2 - 20(1.2)^2 - 5(3)^3 \quad \rightarrow (3)$

4. At $x=3m \Rightarrow y=0=-\frac{1}{2}M_A(3)^2+\frac{1}{6}R_A(3)^3+\frac{10}{3}(1.8)^3-\frac{20}{3}(1.28)^3-\frac{5}{4}(3)^4 \rightarrow (4)$

From equations (3) and (4) $\Rightarrow M_A = 25.4\ KN.m$ $R_A = 46.1\ KN$

Substituting $R_A = 46.1\ KN$ in eq.(1) $R_B = 63.9\ KN$

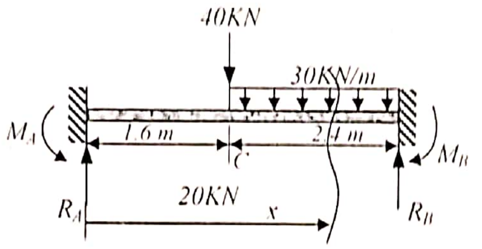
Substituting $R_A$ and $M_A$ in eq.(2) $M_B = 34.9\ KN.m$

Depending on shear force diagram the maximum bending moment occur at one of three points (A,B,at x=1.8m)

$$At\ x = 1.8\ m \Rightarrow M = -24.4 + 46.1(1.8) + 20(0.6) - 15(1.8)^2 = 21.04\ KN.m$$

$$\therefore M_{max} = 34\ KN.m \ \ at\ B$$

$$\sigma_{max} = \frac{M_{max} \cdot Y_{max}}{I} = \frac{34*10^3*100*10^{-3}}{42*10^{-6}} = 81*10^6\ \frac{N}{m^2}$$




Example(2): $EI=14MN/m^2$

Find RA, RB, MA, MB and yc

Sol:

$$\sum Fy=0 \Rightarrow R_A + R_B = 40 + 2.4(30) \rightarrow (1)$$

$$\sum MB = 0$$

$$\Rightarrow -M_A + R_A(4) - 40(2.4) - 30(2.4).\frac{2.4}{2} + M_B = 0 \rightarrow (2)$$

$$EI\ y'' = M = -M_A x^0 + R_A x - -40(x-1.6) - \frac{1}{2}(30)(x-1.6)^2$$

$$EI\ y' = -M_A x + \frac{1}{2}R_A x^2 - -20(x-1.6)^2 - 5(x-1.6)^3 + A$$

$$EI\ y = -\frac{1}{2}M_A x^2 + \frac{1}{6}R_A x^3 - -\frac{20}{3}(x-1.6)^3 - \frac{5}{4}(x-1.6)^4 + Ax + B$$

1. At $x=0 \Rightarrow y=0 \Rightarrow B=0$
2. At $x=0 \Rightarrow y'=0 \Rightarrow A=0$
3. At $x=4m \Rightarrow y'=0=-M_A(4)+\frac{1}{2}R_A(4)^2+20(2.4)^2-5(2.4)^3 \rightarrow (3)$
4. At $x=3m \Rightarrow y=0=-\frac{1}{2}M_A(4)^2+\frac{1}{6}R_A(4)^3+\frac{20}{3}(2.4)^3-\frac{5}{4}(2.4)^4 \rightarrow (4)$

From equations (3) and (4) $\Rightarrow RA=44.1\ KN, MA=42.12\ KN.m$

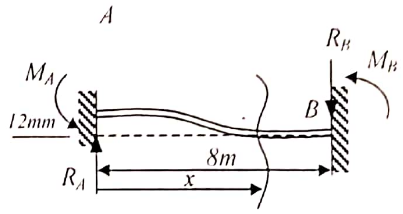
Substituting these values in equation (1) $\Rightarrow RB=67.9\ KN$

Substituting these values in equation (2) $\Rightarrow MB=48.12\ KN.m$

$$\therefore EI\,Yc = -\frac{1}{2}(42.12)(1.6)^2 + \frac{1}{6}(44.1)(1.6)^3$$

$$\therefore Yc = \frac{-23.75}{14*10^3} = -1.7*10^{-3}\,m = -1.7\,mm$$

*Example : 3:* $E = 210\,GN/m^2 \quad I = 90*10^{-6}\,m^4$ *.find* $R_A, R_B, M_A$ *and* $M_B$




$\sum Fy = 0 \quad \Rightarrow R_A = R_B \quad \rightarrow (1)$

$\sum MB = 0 \quad \Rightarrow M_A + M_B = R_A.8 \quad \rightarrow (2)$

$$EIy'' = M_x = -M_A x^0 + R_A x$$

$$EIy' = -M_A x + \frac{1}{2} R_A x^2 + A$$

$$EIy = -\frac{1}{2} M_A x^2 + \frac{1}{6} R_A x^3 + Ax + B$$

*Boundary conditions:*

1. At $x = 0 \quad \Rightarrow y' = 0 \quad \Rightarrow A = 0$
2. At $x = 0 \quad \Rightarrow y = 12\,mm \quad \Rightarrow 90*10^{-6}*210*10^6*0.012 = B = 226.8$
3. At $x = 8m \quad \Rightarrow y' = 0 \quad = -\frac{1}{2} R_A (8)^2 - M_A(8) \quad \Rightarrow R_A = \frac{M_A}{4} \rightarrow (3)$
4. At $x = 8m \quad \Rightarrow y = 0 \Rightarrow 0 = \frac{1}{6} R_A (8)^3 - \frac{1}{2} M_A (8)^2 + 226.8$

$\Rightarrow 85.33 R_A - 32 M_A + 226.8 = 0 \quad \rightarrow (4)$

*From equations (3) and (4)* $\Rightarrow R_A = 5.3156\,KN \qquad M_A = 21.26\,KN.m$

*Substituting these values in (1) and (2) to get :*

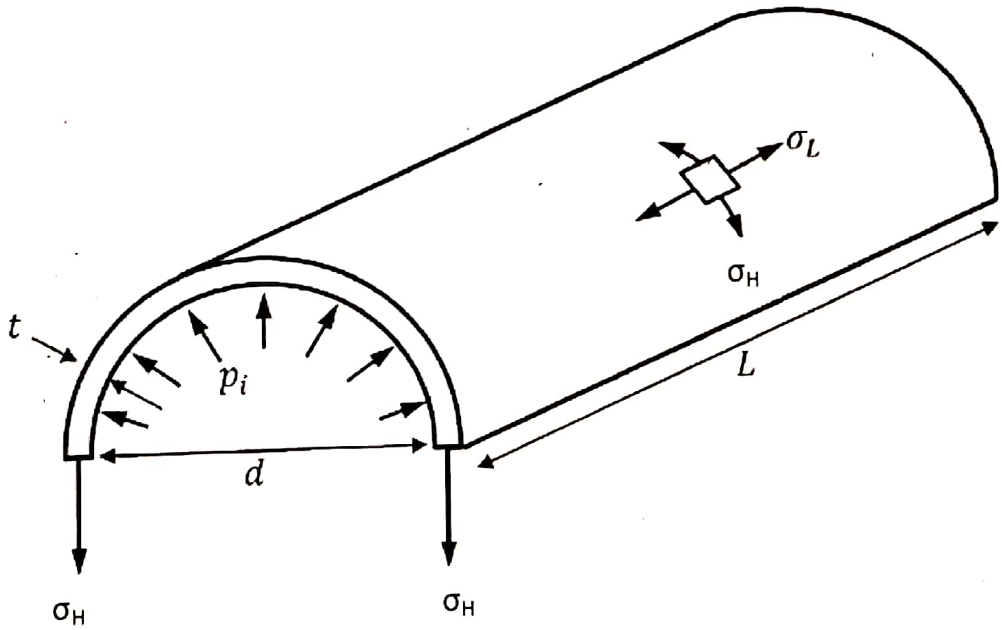
$R_B = 5.3156\,KN \qquad M_B = 21.26\,KN.m$

## Thin Cylinders and spheres

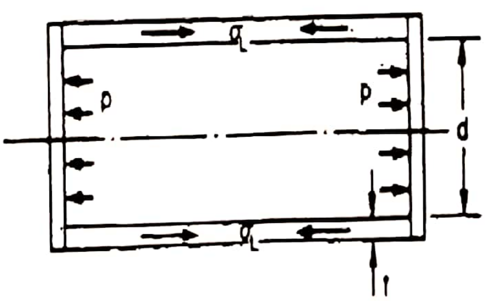
When the thickness of the wall of the cylinderis less than(1/20)of the diameter of cylinder then the cylinder is considered as thin cylinder.Otherwise it is termed as thick cylinder.

Equilibrium of half of the cylinder: $P.d.L = 2\,\sigma_H.\,T.L$

L=Length of the cylinder d= Diameter of cylinder t = thickness of cylinder P= Internal Pressure due to fluid. Circumferential Stress or Hoop Stress ($\sigma_H$). Longitudinal Stress ($\sigma_L$)




$$\sigma_H = \frac{P.d}{2.t}$$




**Longitudinal stress:** Consider now the cylinder as shown. Total force on the end of the cylinder owing to internal pressure: pressure x area = $p \times \pi d^2/4 = \sigma_L.\pi.D.t$

$$\sigma_L = \frac{P.d}{4.t}$$

**Change in Length:**

The change in length of the cylinder may be determined from the longitudinal strain, i.e. neglecting the radial stress.

$$\text{Longitudinal strain} = \frac{1}{E}[\sigma_L - \nu\sigma_H]$$

and change in length = longitudinal strain × original length

$$= \frac{1}{E}[\sigma_L - \nu\sigma_H]L$$

$$= \frac{pd}{4tE}[1 - 2\nu]L$$

**Change in Diameter:**

$$\epsilon_H = \frac{1}{E}(\sigma_H - \nu\sigma_L)$$

$$\frac{\pi(D + \Delta D)}{\pi D} = \frac{1}{E}\left(\frac{PD}{2t} - \nu\frac{PD}{4t}\right)$$

$$\Delta D = \frac{PD^2}{4tE}(2 - \nu)$$

Change in Internal volume:

$$V = \frac{\pi}{4}D^2L \quad \rightarrow \Delta V = \frac{\pi}{4}(D^2\Delta L + L.D.\Delta D)$$

$$\Delta V = \frac{\pi}{4}\,D^2L\left(\frac{\Delta L}{L} + 2\frac{\Delta D}{D}\right)$$

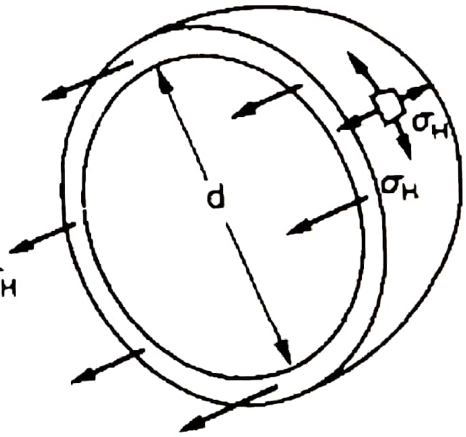
$$\Delta V = V\left[\frac{PD}{4tE}(1-2\nu) + \frac{PD}{4tE}(2-\nu)\right]$$

$$\Delta V = V\left[\frac{PD}{4tE}(5-4\nu)\right]$$

**Thin shperes under internal pressure:**

Equilibrium of half of the sphere:

Total force on the end of the cylinder owing to internal pressure: pressure x area = p x $\pi d^2/4 = \sigma_H.\pi.D.t$




$$\sigma_H = \frac{P.d}{4.t}$$

$$\epsilon_H = \frac{1}{E}(\sigma_H - \nu\sigma_L)$$

$$\frac{\pi(D+\Delta D)}{\pi D} = \frac{1}{E}\left(\frac{PD}{4t} - \nu\frac{PD}{4t}\right)$$

$$\Delta D = \frac{PD^2}{4tE}(1-\nu)$$

$$V = \frac{\pi}{6}D^3 \quad \rightarrow \Delta V = \frac{\pi}{6}(3D^2\Delta D)$$

$$\Delta V = V.\left(3\frac{\Delta D}{D}\right) = \Delta D = \frac{3PD}{4tE}(1-\nu)$$

### Vessels subjected to fluid pressure :

If a fluid is used as the pressurisation medium the fluid itself will change in volume as pressure is increased and this must be taken into account when calculating the amount of fluid which must be pumped into the cylinder in order to raise the pressure by a specified amount, the cylinder being initially full of fluid at atmospheric pressure. Now the bulk modulus of a fluid is defined as follows:

$$\text{bulk modulus } K = \frac{\text{volumetric stress}}{\text{volumetric strain}}$$

where, in this case, volumetric stress = pressure $p$

and

$$\text{volumetric strain} = \frac{\text{change in volume}}{\text{original volume}} = \frac{\delta V}{V}$$

$$\therefore \quad K = \frac{p}{\delta V/V} = \frac{pV}{\delta V}$$

i.e.

$$\textbf{change in volume of fluid under pressure} = \frac{pV}{K}$$

The extra fluid required to raise the pressure must, therefore, take up this volume together with the increase in internal volume of the cylinder itself.

∴ extra fluid required to raise *cylinder* pressure by $p$

$$= \frac{pd}{4tE}[5 - 4\nu]V + \frac{pV}{K}$$

Similarly, for *spheres*, the extra fluid required is

$$= \frac{3pd}{4tE}[1 - \nu]V + \frac{pV}{K}$$

**Example**: (a) A sphere, 1m internal diameter and 6mm wall thickness, is to be pressure-tested for safety purposes with water as the pressure medium. Assuming that the sphere is initially filled with water at atmospheric pressure, what extra volume of water is required to be pumped in to produce a pressure of 3 $MN/m^2$ gauge? For water, K = 2.1 $GN/m^2$. (b) The sphere is now placed in service and filled with gas until there is a volume change of 72 $\times 10^{-6}$ $m^3$. Determine the pressure exerted by the gas on the walls of the sphere. (c) To what value can the gas pressure be increased before failure occurs according to the maximum principal stress theory of elastic failure? For the material of the sphere E = 200 $GN/m^2$, v = 0.3 and the yield stress $\sigma_y$, in simple tension = 280 $MN/m^2$.

*Solution*

(a) Bulk modulus $K = \dfrac{\text{volumetric stress}}{\text{volumetric strain}}$

Now volumetric stress = pressure $p = 3\ \text{MN/m}^2$

and volumetric strain = change in volume ÷ original volume

i.e. $$K = \frac{p}{\delta V/V}$$

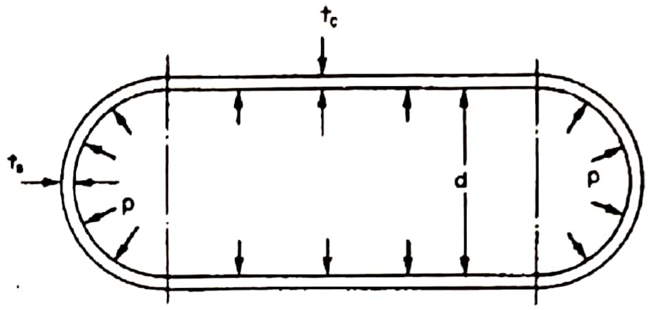
$\therefore$ $$\text{change in volume of water} = \frac{pV}{K} = \frac{3\times10^6}{2.1\times10^9}\times\frac{4\pi}{3}(0.5)^3$$

$$= \mathbf{0.748\times10^{-3}\ m^3}$$

(b) From eqn. (9.9) the change in volume is given by

### *Cylindrical vessel with hemispherical ends:*

Consider now the vessel as shown in which the wall thickness of the cylindrical and hemispherical portions may be different (this is sometimes necessary since the hoop stress in the cylinder is twice that in a sphere of the same radius and wall thickness). For the purpose of the calculation the internal diameter of both portions is assumed equal. From the preceding sections the following formulae are known to apply.




*(a) For the cylindrical portion:*

$$\text{hoop or circumferential stress} = \sigma_{H_c} = \frac{pd}{2t_c}$$

$$\text{longitudinal stress} = \sigma_{L_c} = \frac{pd}{4t_c}$$

$$\text{hoop or circumferential strain} = \frac{1}{E}[\sigma_{H_c} - \nu\sigma_{L_c}] = \frac{pd}{4t_c E}[2-\nu]$$

*(b) For the hemispherical ends:*

$$\text{hoop stress} = \sigma_{H_s} = \frac{pd}{4t_s}$$

$$\text{hoop strain} = \frac{1}{E}[\sigma_{H_s} - \nu\sigma_{H_s}] = \frac{pd}{4t_s E}[1-\nu]$$

Thus equating the two strains in order that there shall be no distortion of the junction

$$\frac{pd}{4t_c E}[2-\nu] = \frac{pd}{4t_s E}[1-\nu] \quad \text{i.e.} \quad \frac{t_s}{t_c} = \frac{(1-\nu)}{(2-\nu)}$$

$$\delta V = \frac{3pd}{4tE}(1-\nu)V$$

$$\therefore \quad 72 \times 10^{-6} = \frac{3p \times 1 \times \frac{4}{3}\pi(0.5)^3(1-0.3)}{4 \times 6 \times 10^{-3} \times 200 \times 10^9}$$

$$\therefore \quad p = \frac{72 \times 10^{-6} \times 4 \times 6 \times 200 \times 10^6 \times 3}{3 \times 4\pi(0.5)^3 \times 0.7}$$

$$= 314 \times 10^3 \text{ N/m}^2 = \mathbf{314\ kN/m^2}$$

(c) The maximum stress set up in the sphere will be the hoop stress,

i.e. $$\sigma_1 = \sigma_H = \frac{pd}{4t}$$

Now, according to the maximum principal stress theory failure will occur when the maximum principal stress equals the value of the yield stress of a specimen subjected to simple tension,

i.e. when $$\sigma_1 = \sigma_y = 280 \text{ MN/m}^2$$

Thus $$280 \times 10^6 = \frac{pd}{4t}$$

$$p = \frac{280 \times 10^6 \times 4 \times 6 \times 10^{-3}}{1}$$

$$= 6.72 \times 10^6 \text{ N/m}^2 = \mathbf{6.7\ MN/m^2}$$

The sphere would therefore yield at a pressure of 6.7 $\text{MN/m}^2$.

With the normally accepted value of Poisson's ratio for general steel work of 0.3, the thickness ratio becomes :

$$\frac{t_s}{t_c} = \frac{0.7}{1.7}$$

i.e. the thickness of the cylinder walls must be approximately 2.4 times that of the hemispherical ends for no distortion of the junction to occur.

**Example**: A cylinder has an internal diameter of 230 mm, has walls 5 mm thick and is 1 m long. It is found to change in internal volume by 12.0 x $m^3$ when filled with a liquid at a pressure p . If E = 200GN/$m^2$ and γ= 0.25, and assuming rigid end plates, determine:

(a) the values of hoop and longitudinal stresses;

(c) the necessary change in pressure p to produce a further increase in internal volume of (longitudinal) are assumed; 15 %. The liquid may be assumed incompressible.

1. Determine the change in volume of a thin cylinder of original volume 65.5 **x** 10- m3 and length 1.3 m if its wall thickness is 6 mm and the internal pressure **14** bar (1.4 $MN/m^2$). For the ylinder material E = 210 $GN/m^2$; $\nu = 0.3$. ans:17.5 **x** 10-6$m^3$.]

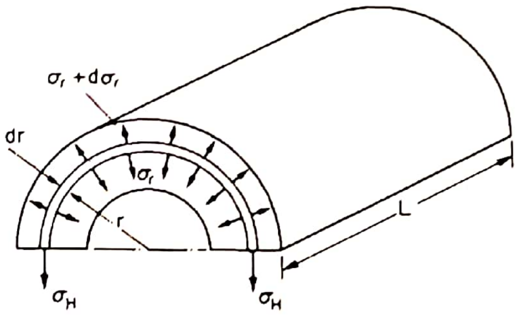
2. What must be the wall thickness of a thin spherical vessel of diameter 1 m if it is to withstand an internal pressure of 70 bar (7 $MN/m^2$) and the hoop stresses are limited to 270 $MN/m^2$

3. A steel cylinder 1 m long, of 150mm internal diameter and plate thickness 5mm, is subjected to an internal pressure of 70bar (7 $MN/m^2$); the increase in volume owing to the pressure is 16.8 x m3. Find the values of Poisson's ratio and the modulus of rigidity. Assume E = 210$GN/m^2$.

4. A spherical vessel of 1.7m diameter is made from 12mm thick plate, and it is to be subject4 to a hydraulic test. Determine the additional volume of water which it is necessary to pump into the vessel, which the vessel is initially just filled with water, in order to raise the pressure to the proof pressure of 116 bar (1 1.6 $MN/m^2$). The bulk modulus of water is 2.9 $GN/m^2$. For the material of the vessel , E = 200 $GN/m^2$, v = 0.3. ans:26.14 x10-6 $m^3$

**Thick cylinders:**

**Lame theory:**

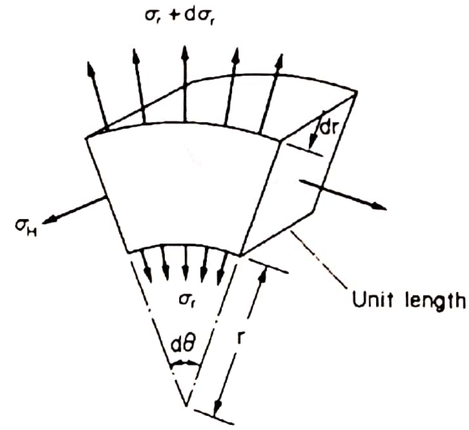



Consider the thick cylinder as shown. The stresses acting on an element of unit length at radius rare as shown in Fig. the radial stress increasing from σ, to σ, + dσ, over the element thickness dr (all stresses are assumed tensile),

For radial equilibrium of the element:

$$\sum F_r = \quad (\sigma_r + d\sigma_r)(r + dr)d\theta \times 1 - \sigma_r \times rd\theta \times 1 = 2\sigma_H \times dr \times 1 \times \sin\frac{d\theta}{2}$$

For small angles: $\sin\frac{d\theta}{2} \doteq \frac{d\theta}{2}$ radian




Therefore, neglecting second-order small quantities,

$$rd\sigma_r + \sigma_r dr = \sigma_H dr \qquad \sigma_r + r\frac{d\sigma_r}{dr} = \sigma_H$$

Or: $$\sigma_H - \sigma_r = r\frac{d\sigma_r}{dr} \qquad (1)$$

Assuming now that plane sections remain plane, i.e. the longitudinal strain .zL is constant across the wall of the cylinder:

$$\varepsilon_L = \frac{1}{E}[\sigma_L - \nu\sigma_r - \nu\sigma_H \quad = \frac{1}{E}[\sigma_L - \nu(\sigma_r + \sigma_H)] = \text{constant}$$

It is also assumed that the longitudinal stress $\sigma_L$ is constant across the cylinder walls at points remote from the ends

$$\sigma_r + \sigma_H = \text{constant} = 2A \text{ (say)} \qquad (2)$$

Substituting in (1) for $\sigma_r$: $$2A - \sigma_r - \sigma_r = r\frac{d\sigma_r}{dr}$$

Multiplying through by $r$ and rearranging, $$2\sigma_r r + r^2\frac{d\sigma_r}{dr} - 2Ar = 0$$

$$\frac{d}{dr}(\sigma_r r^2 - Ar^2) = 0$$

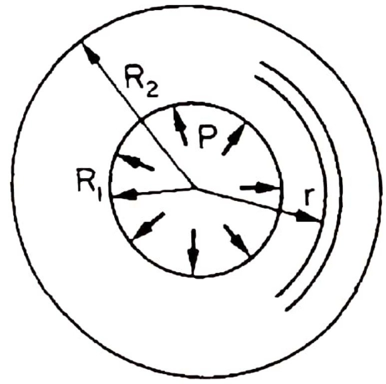
Therefore, integrating, $$\sigma_r r^2 - Ar^2 = \text{constant} = -B \text{ (say)} \qquad \sigma_r = A - \frac{B}{r^2}$$

And from equation (2) : $$\sigma_H = A + \frac{B}{r^2}$$

**Thick cylinder - internal pressure only:**

Consider now the thick cylinder as shown subjected to an internal pressure P, the external pressure being zero:

$$\text{At } r = R_1 \quad \sigma_r = -P \quad \text{and} \quad \text{at } r = R_2 \quad \sigma_r = 0$$




The internal pressure is considered as a negative radial stress since it will produce a radial compression (i.e. thinning) of the cylinder walls and the normal stress convention takes compression as negative.

Substituting the above conditions in radial stress equation:

$$-P = A - \frac{B}{R_1^2} \qquad \text{And} \qquad 0 = A - \frac{B}{R_2^2}$$

$$A = \frac{PR_1^2}{(R_2^2 - R_1^2)} \quad \text{and} \quad B = \frac{PR_1^2 R_2^2}{(R_2^2 - R_1^2)}$$

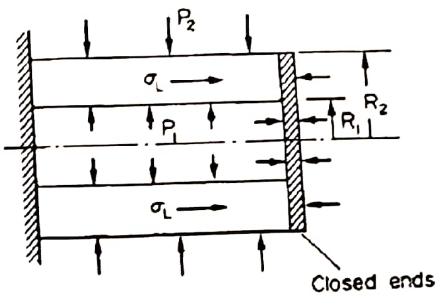
$$\textbf{radial stress } \sigma_r = A - \frac{B}{r^2} = \frac{PR_1^2}{(R_2^2 - R_1^2)}\left[1 - \frac{R_2^2}{r^2}\right]$$

**where $k$ is the diameter ratio $D_2/D_1 = R_2/R_1$**

**and** $$\text{hoop stress } \sigma_H = \frac{PR_1^2}{(R_2^2 - R_1^2)}\left[1 + \frac{R_2^2}{r^2}\right]$$

$$= \frac{PR_1^2}{(R_2^2 - R_1^2)}\left[\frac{r^2 + R_2^2}{r^2}\right] = P\left[\frac{(R_2/r)^2 + 1}{k^2 - 1}\right]$$

**Longitudinal stress:** Consider now the cross-section of a thick cylinder with closed ends subjected to an internal pressure $P^{l}$ and an external pressure $P_2$,




For horizontal equilibrium:

$$P_1 \times \pi R_1^2 - P_2 \times \pi R_2^2 = \sigma_L \times \pi(R_2^2 - R_1^2)$$

where $\sigma_L$ is the longitudinal stress set up in the cylinder walls,

$$\text{longitudinal stress } \sigma_L = \frac{P_1 R_1^2 - P_2 R_2^2}{R_2^2 - R_1^2}$$

$\sigma_L = A$

**Cbange of cylinder dimensions: (a) change in diameter**

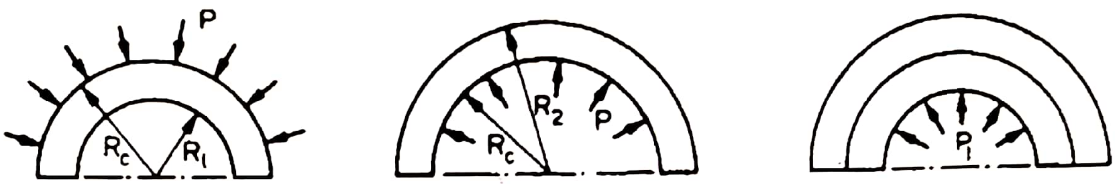
$$\varepsilon_H = \frac{1}{E}[\sigma_H - v\sigma_r - v\sigma_L]$$

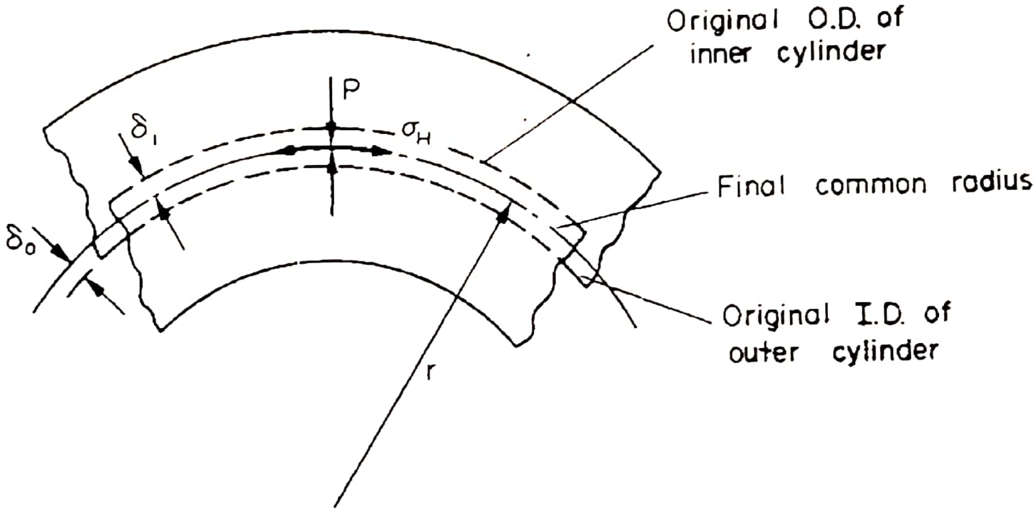
$$\Delta D = \frac{2r}{E}[\sigma_H - v\sigma_r - v\sigma_L]$$

**(b) Change of length:**

$$\Delta L = \frac{L}{E}[\sigma_L - \sigma v_r - v\sigma_H]$$

**Compound cylinders:**




(a) *shrinkage-internal cylinder:*

At $r = R_1$, $\sigma_r = 0$

At $r = R_c$, $\sigma_r = -p$ (compressive since it tends to reduce the wall thickness)

condition (b) *shrinkage – external cylinder:*

At $r = R_2$, $\sigma_r = 0$

At $r = R_c$, $\sigma_r = -p$

condition (c) *internal pressure – compound cylinder:*

At $r = R_2$, $\sigma_r = 0$

At $r = R_1$, $\sigma_r = -P_1$

**Shrinkage or interference allowance:**

since circumferential strain = diametral strain

$$\text{circumferential strain at radius } r \text{ on outer cylinder} = \frac{2\delta_o}{2r} = \frac{\delta_o}{r} = \varepsilon_{H_o}$$

$$\text{circumferential strain at radius } r \text{ on inner cylinder} = \frac{2\delta_i}{2r} = \frac{\delta_i}{r} = -\varepsilon_{H_i}$$

(negative since it is a *decrease* in diameter).

$$\text{Total interference or shrinkage} = \delta_o + \delta_i = r\varepsilon_{H_o} + r(-\varepsilon_{H_i})$$

$$= (\varepsilon_{H_o} - \varepsilon_{H_i})r$$

Now assuming open ends, i.e. $\sigma_L = 0$,

$$\varepsilon_{H_o} = \frac{\sigma_{H_o}}{E_1} - \frac{\nu_1}{E_1}(-p) \qquad \text{since } \sigma_{r_o} = -p$$

and

$$\varepsilon_{H_i} = \frac{\sigma_{H_i}}{E_2} - \frac{\nu_2}{E_2}(-p) \qquad \text{since } \sigma_{r_i} = -p$$

Therefore total interference or shrinkage allowance $= \left[\frac{1}{E_1}(\sigma_{H_o} + \nu_1 p) - \frac{1}{E_2}(\sigma_{H_i} + \nu_2 p)\right]r$

Generally, however, the tubes are of the same material.

$$\textbf{Shrinkage allowance} = \frac{r}{E}(\sigma_{H_o} - \sigma_{H_i})$$

*Example1*:

A thin cylinder 75 mm internal diameter, 250 mm long with walls 2.5 mm thick is subjected to an internal pressure of 7 $MN/m^2$. Determine the change in internal diameter and the change in length.

If, in addition to the internal pressure, the cylinder is subjected to a torque of 200 N m, find the magnitude and nature of the principal stresses set up in the cylinder. $E = 200\ GN/m^2$. $\nu = 0.3$.

*Example 1:* A thick cylinder of 100 mm internal radius and 150 mm external radius is subjected to an internal pressure of 60 MN/m$^2$ and an external pressure of 30 MN/m$^2$. Determine the hoop and radial stresses at the inside and outside of the cylinder together with the longitudinal stress if the cylinder is assumed to have closed ends.

**Solution:**

at $r = 0.1$ m, $\sigma_r = -60\,\text{MN/m}^2$ at $r = 0.15$ m, $\sigma_r = -30\,\text{MN/m}^2$

$$-60 = A - 100B \qquad -30 = A - 44.5B$$

B = 0.54 and A = -6

$$\sigma_H = A + \frac{B}{r^2} = -6 + 0.54 \times 100 = \mathbf{48\,MN/m^2}$$

and at $r = 0.15$ m,

$$\sigma_H = -6 + 0.54 \times 44.5 = -6 + 24 = \mathbf{18\,MN/m^2}$$

From eqn. (10.7) the longitudinal stress is given by

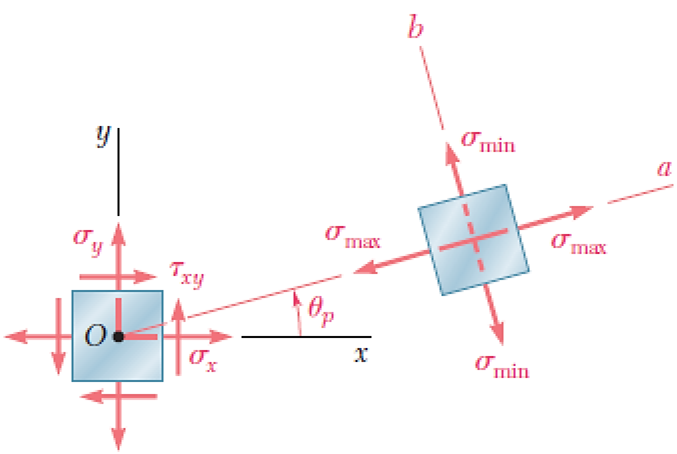
$$\sigma_L = \frac{P_1R_1^2 - P_2R_2^2}{(R_2^2 - R_1^2)} = \frac{(60 \times 0.1^2 - 30 \times 0.15^2)}{(0.15^2 - 0.1^2)}$$

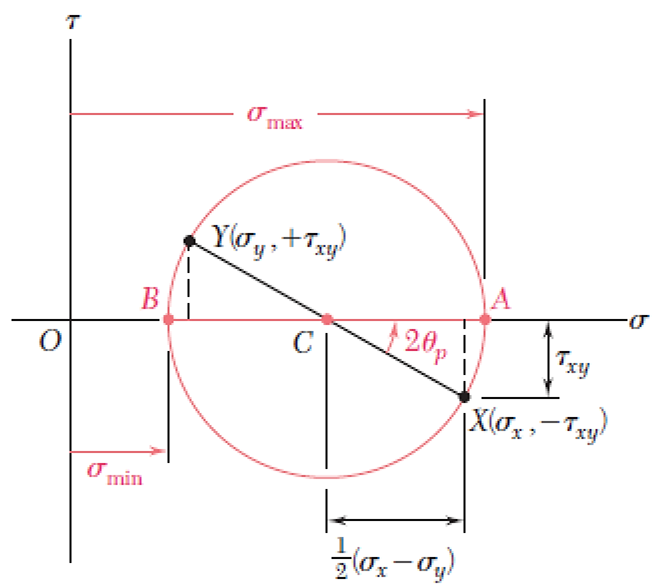
$$= \frac{10^2(60 - 30 \times 2.25)}{1.25 \times 10^2} = -\mathbf{6\,MN/m^2} \quad \text{i.e. compressive}$$

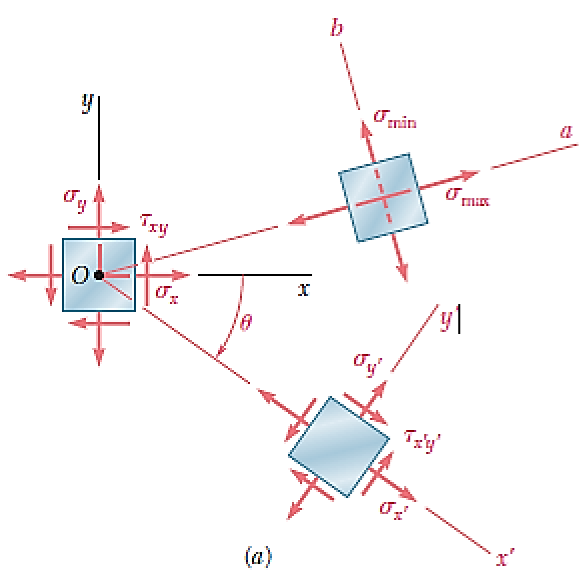
*Example* An external pressure of 10 MN/m$^2$ is applied to a thick cylinder of internal diameter 160 mm and external diameter 320 mm. If the maximum hoop stress permitted on the inside wall of the cylinder is limited to 30 MN/m$^2$, what maximum internal pressure can be applied assuming the cylinder has closed ends? What will be the change in outside diameter when this pressure is applied? E = 207 GN/m$^2$, v = 0.29.

## Combined stresses:

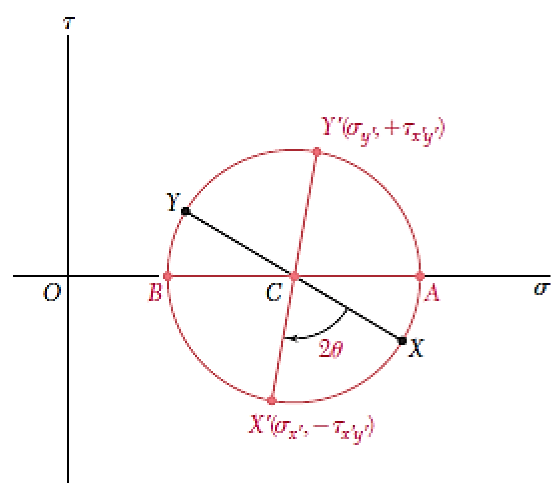
The circle used in the preceding section to derive some of the basic formulas relating to the transformation of plane stress was first introduced by the German engineer Otto Mohr (1835–1918) and is known as *Mohr's circle* for plane stress. This method is based on simple geometric considerations and does not require the use of specialized formulas. While originally designed for graphical solutions.



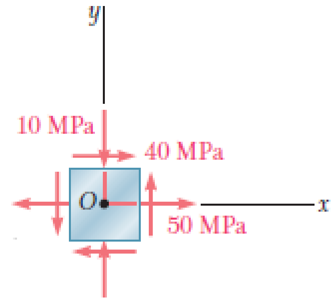




(a)

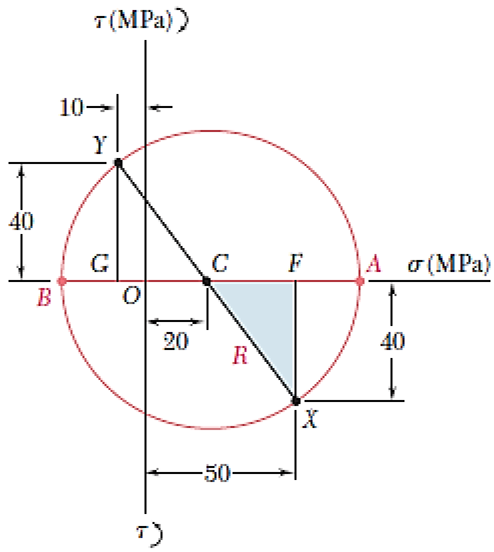


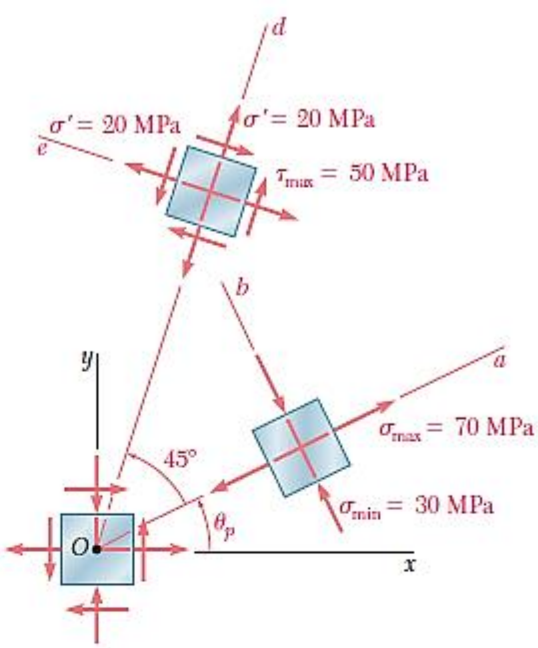
**Example1:** For the state of plane stress already considered as shown in figure, **(a)** Construct Mohr's circle, **(b)** Determine the principal stresses, **(c)** Determine the maximum shearing stress and the corresponding normal stress.

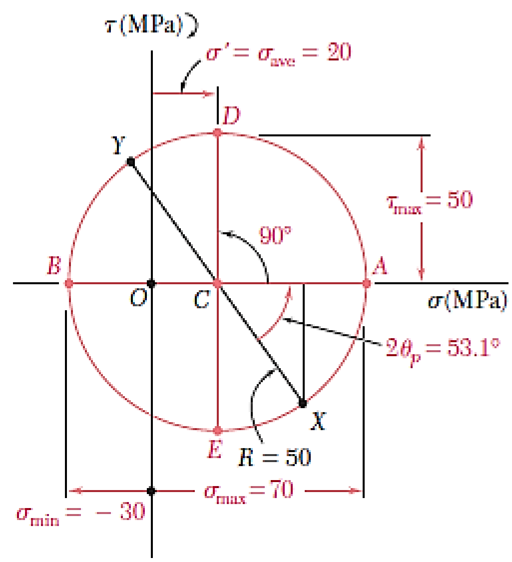



(a)

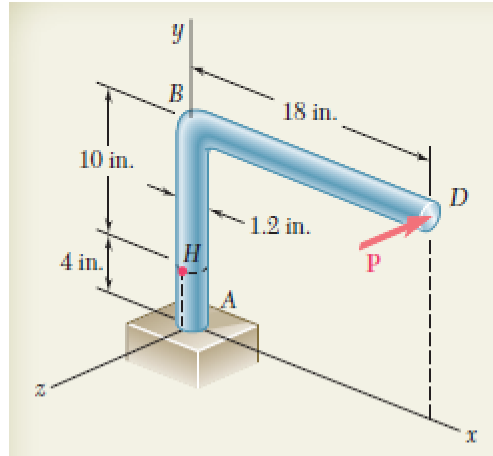
**Solution:**

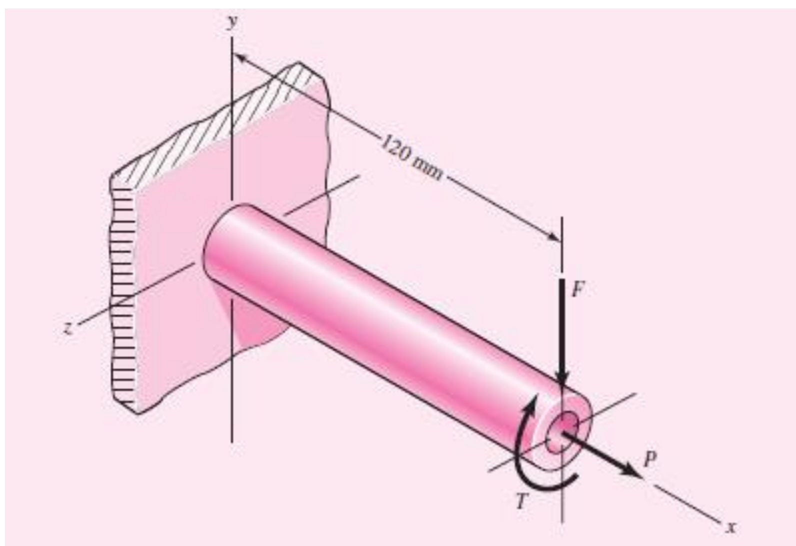


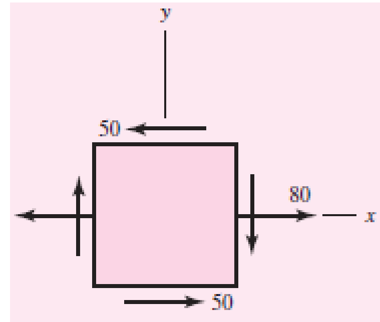



**Example2:** single horizontal force P of magnitude 150 lb is applied to end D of lever ABD. Knowing that portion AB of the lever has a diameter of 1.2 in., determine (a) the normal and shearing stresses on an element located at point H and having sides parallel to the x and y axes, (b) the principal planes and the principal stresses at point H.

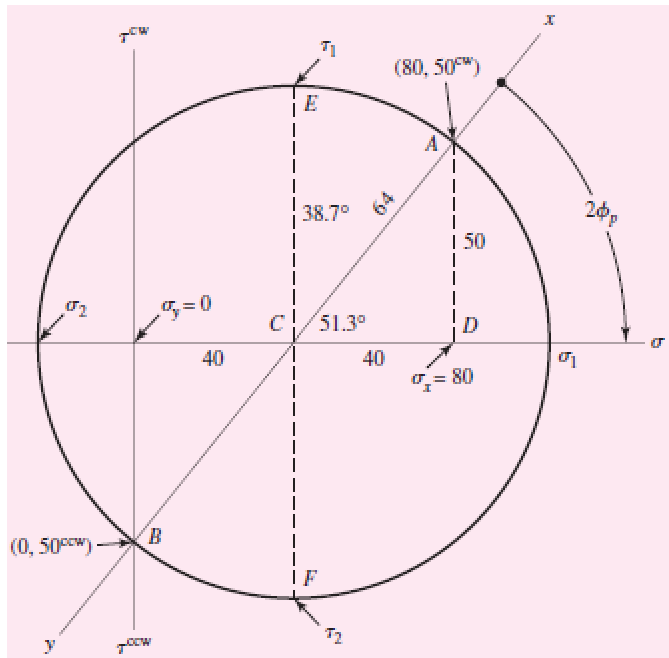


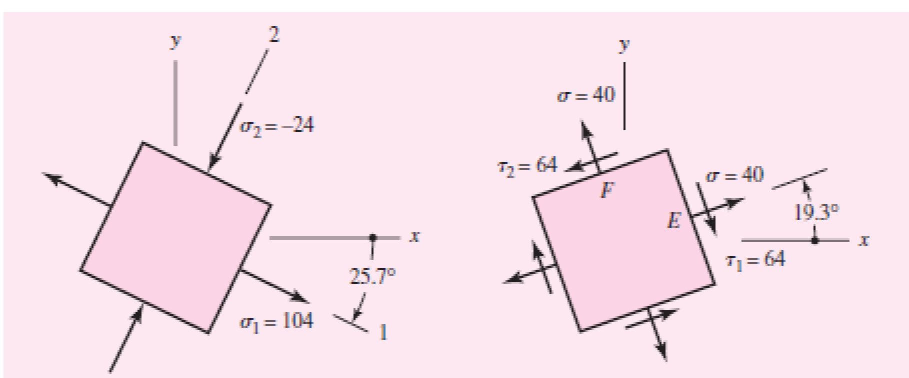



**Example3:** A stress element has $\sigma_x$ = 80 MPa and $\tau_{xy}$ = 50 MPa cw, as shown in Figure. Using Mohr's circle, find the principal stresses and directions, and show these on a stress element correctly aligned with respect to the xy coordinates. Draw another stress element to show $\tau_1$ and $\tau_2$, find the corresponding normal stresses, and label the drawing completely.



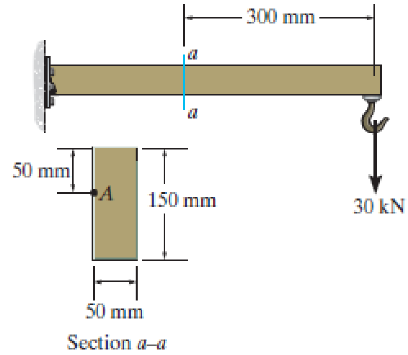

Solution:



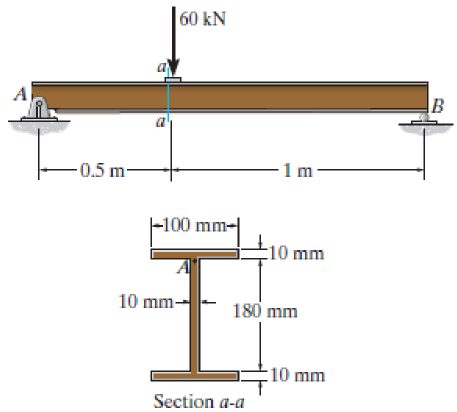



H.w:

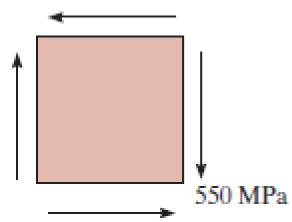
1. Determine the principal stress developed at point A on the cross section of the beam at section a–a.



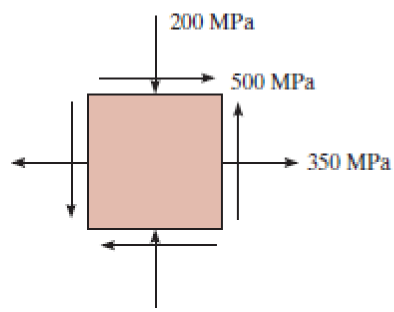

2. Determine the maximum in-plane shear stress developed at point A on the cross section of the beam at section a–a, which is located just to the left of the 60-kN force. Point A is just below the flange.




3. Determine the equivalent state of stress if an element is oriented 25° counterclockwise from the element shown.




4. Determine the principal stress, the maximum in-plane shear stress, and average normal stress. Specify the orientation of the element in each case.



## 2-1. Failure Theories

Unfortunately, there is no universal theory of failure for the general case of material properties and stress state. Instead, over the years several hypotheses have been formulated and tested, leading to today's accepted practices. Being accepted, we will characterize these "practices" as *theories* as most designers do.

Structural metal behavior is typically classified as being ductile or brittle, although under special situations, a material normally considered ductile can fail in a brittle manner. Ductile materials are normally classified such that $\varepsilon_f \geq 0.05$ and have an identifiable yield strength that is often the same in compression as in tension ($S_{yt} = S_{yc} = S_y$). Brittle materials, $\varepsilon_f < 0.05$, do not exhibit an identifiable yield strength, and are typically classified by ultimate tensile and compressive strengths, $S_{ut}$ and $S_{uc}$, respectively (where $S_{uc}$ is given as a positive quantity). The generally accepted theories are:

**Ductile materials** (yield criteria)

- Maximum shear stress (MSS)
- Distortion energy (DE)
- Ductile Coulomb-Mohr (DCM)

**Brittle materials** (fracture criteria)

- Maximum normal stress (MNS)
- Brittle Coulomb-Mohr (BCM)
- Modified Mohr (MM)

## 2-2. Maximum-Shear-Stress Theory for Ductile Materials (MSS)

The *maximum-shear-stress theory* predicts that *yielding begins whenever the maximum shear stress in any element equals or exceeds the maximum shear stress in a tension-test specimen of the same material when that specimen begins to yield*.

The maximum-shear-stress theory predicts yielding when

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{S_y}{2n}$$

Where $S_y$ is the yielding stress, and $n$ is the factor of safety. Note that this implies that the yield strength in shear is given by

$$S_{sy} = 0.5S_y$$

The MSS theory is also referred to as the *Tresca* or *Guest theory*. It is an acceptable theory but conservative predictor of failure; and since engineers are conservative by nature, it is quite often used.

## 2-3. Distortion-Energy Theory for Ductile Materials (DE)

The distortion-energy theory predicts that yielding occurs when the distortion strain energy per unit volume reaches or exceeds the distortion strain energy per unit volume for yield in simple tension or compression of the same material.

The distortion-energy theory is also called:
• The von Mises or von Mises–Hencky theory
• The shear-energy theory
• The octahedral-shear-stress theory

The distortion-energy theory predicts yielding when

$$\sigma' = \frac{S_y}{n}$$

where σ′ is usually called the *von Mises stress,* named after Dr. R. von Mises, who contributed to the theory; and

$$\sigma' = \left(\sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\tau_{xy}^2\right)^{1/2}$$

The shear yield strength predicted by the distortion-energy theory is

$$S_{sy} = 0.577S_y$$

**EXAMPLE 2-1**

A hot-rolled steel has a yield strength of $S_{yt} = S_{yc} = 100$ kpsi and a true strain at fracture of $\varepsilon_f = 0.55$. Estimate the factor of safety for the following principal stress states:

(*a*) 70, 70, 0 Mpa
(*b*) 30, 70, 0 Mpa.
(*c*) 0, 70, −30 Mpa.
(*d*) 0, −30, −70 Mpa.
(*e*) 30, 30, 30 Mpa.

**Solution**

## 2-4. Coulomb-Mohr Theory for Ductile Materials (DCM)

A variation of Mohr's theory, called the *Coulomb-Mohr theory* or the *internal-friction theory*.

Not all materials have compressive strengths equal to their corresponding tensile values. For example, the yield strength of magnesium alloys in compression may be as little as 50 percent of their yield strength in tension. The ultimate strength of gray cast irons in compression varies from 3 to 4 times greater than the ultimate tensile strength. So, this theory can be used to predict failure for materials whose strengths in tension and compression are not equal; this is can be expressed as a design equation with a factor of safety, *n*, as

$$\frac{\sigma_1}{S_t} - \frac{\sigma_3}{S_c} = 1$$

where either yield strength or ultimate strength can be used.
The torsional yield strength occurs when $\tau_{max} = S_{sy}$ ; then

$$S_{sy} = \frac{S_{yt} S_{yc}}{S_{yt} + S_{yc}}$$

**EXAMPLE 2-2**

A 25-mm-diameter shaft is statically torqued to 230 N·m. It is made of cast 195-T6 aluminum, with a yield strength in tension of 160 MPa and a yield strength in compression of 170 MPa. It is machined to final diameter. Estimate the factor of safety of the shaft.

**Solution**

The maximum shear stress is given by

$$\tau = \frac{16T}{\pi d^3} = \frac{16(230)}{\pi \left[25\left(10^{-3}\right)\right]^3} = 75\left(10^6\right)\,\text{N/m}^2 = 75\,\text{MPa}$$

The two nonzero principal stresses are 75 and −75 MPa, making the ordered principal stresses $\sigma_1 = 75$, $\sigma_2 = 0$, and $\sigma_3 = -75$ MPa. From Eq. (2–6), for yield,

$$n = \frac{1}{\sigma_1/S_{yt} - \sigma_3/S_{yc}} = \frac{1}{75/160 - (-75)/170} = 1.10$$

Alternatively, from Eq. (2–7),

$$S_{sy} = \frac{S_{yt}S_{yc}}{S_{yt} + S_{yc}} = \frac{160(170)}{160 + 170} = 82.4\,\text{MPa}$$

and $\tau_{max} = 75$ MPa. Thus,

$$n = \frac{S_{sy}}{\tau_{max}} = \frac{82.4}{75} = 1.10$$

**2-5. Maximum-Normal-Stress Theory for Brittle Materials (MNS)**

The maximum-normal-stress (MNS) theory states that *failure occurs whenever one of the three principal stresses equals or exceeds the strength*. Again we arrange the principal stresses for a general stress state in the ordered form $\sigma_1 \geq \sigma_2 \geq \sigma_3$. This theory then predicts that failure occurs whenever

$$\sigma_1 \geq S_{ut} \qquad \text{or} \qquad \sigma_3 \leq -S_{uc}$$

where $S_{ut}$ and $S_{uc}$ are the ultimate tensile and compressive strengths, respectively, given as positive quantities.

## 2-6. Modifications of the Mohr Theory for Brittle Materials

We will discuss two modifications of the Mohr theory for brittle materials: the Brittle- Coulomb-Mohr (BCM) theory and the modified Mohr (MM) theory. The equations provided for the theories will be restricted to plane stress and be of the design type incorporating the factor of safety.

### ***Brittle-Coulomb-Mohr*** **(*BCM*)**

$$\sigma_A = \frac{S_{ut}}{n} \qquad \sigma_A \ge \sigma_B \ge 0$$

$$\frac{\sigma_A}{S_{ut}} - \frac{\sigma_B}{S_{uc}} = \frac{1}{n} \qquad \sigma_A \ge 0 \ge \sigma_B$$

$$\sigma_B = -\frac{S_{uc}}{n} \qquad 0 \ge \sigma_A \ge \sigma_B$$

### ***Modified Mohr*** **(*MM*)**

$$\sigma_A = \frac{S_{ut}}{n} \qquad \sigma_A \ge \sigma_B \ge 0$$

$$\sigma_A \ge 0 \ge \sigma_B \quad \text{and} \quad \left|\frac{\sigma_B}{\sigma_A}\right| \le 1$$

$$\frac{(S_{uc} - S_{ut})\,\sigma_A}{S_{uc}S_{ut}} - \frac{\sigma_B}{S_{uc}} = \frac{1}{n} \qquad \sigma_A \ge 0 \ge \sigma_B \quad \text{and} \quad \left|\frac{\sigma_B}{\sigma_A}\right| > 1$$

$$\sigma_B = -\frac{S_{uc}}{n} \qquad 0 \ge \sigma_A \ge \sigma_B$$

### EXAMPLE 2-4

Consider the wrench in Ex. (2–3), Fig. (2–1), as made of cast iron, machined to dimension. The force *F* required to fracture this part can be regarded as the strength of the component part. If the material is cast iron, the tensile ultimate strength is 31 kpsi and the compressive ultimate strength is 109 kpsi, find the force *F* with
(a) Coulomb-Mohr failure model (*b*) Modified Mohr failure model.

## 2-7. Selection of Failure Criteria

For ductile behavior the preferred criterion is the distortion-energy theory, although some designers also apply the maximum-shear-stress theory because of its simplicity and conservative nature. In the rare case when $S_{yt} \neq S_{yc}$ , the ductile Coulomb-Mohr method is employed.

For brittle behavior, the original Mohr hypothesis, constructed with tensile, compression, and torsion tests, with a curved failure locus is the best hypothesis we have. However, the difficulty of applying it without a computer leads engineers to choose modifications, namely, Coulomb Mohr, or modified Mohr. Figure (2–2) provides a summary flowchart for the selection of an effective procedure for analyzing or predicting failures from static loading for brittle or ductile behavior.

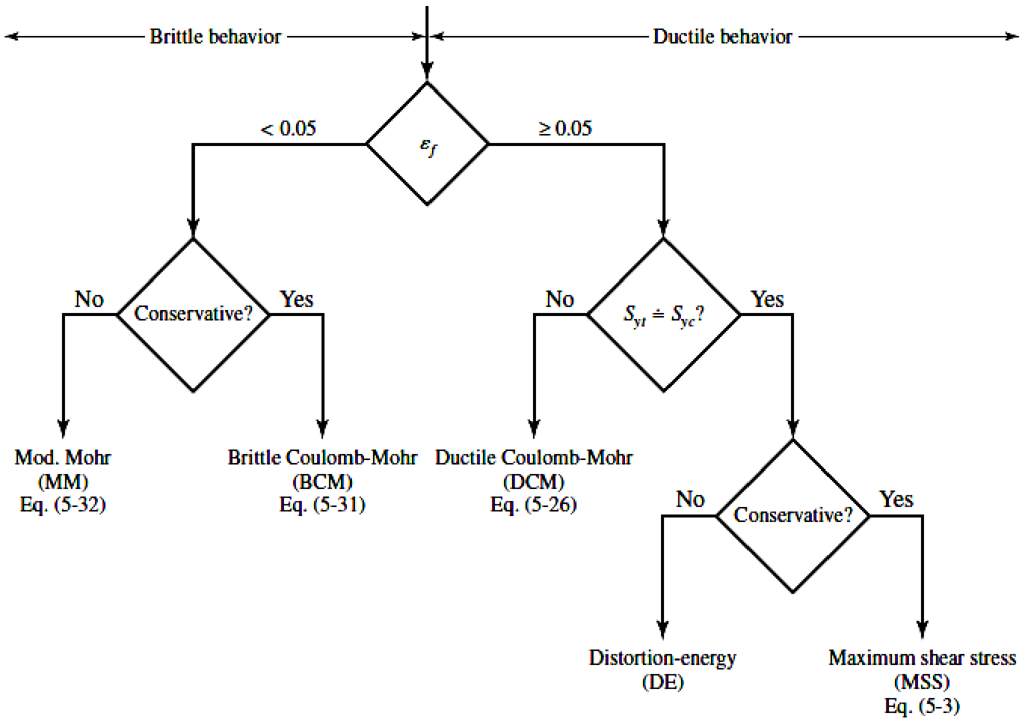



**Figure (2-2)**

Failure theory selection flowchart.

## Homework

(1) The cantilevered bar shown in the figure is made of AISI 1006 cold-drawn steel with ($S_y$ = 280 MPa) and is loaded by the forces $F$ = 0.55 kN, $P$ = 8 kN, and $T$ = 30 N·m. Compute the factor of safety, based upon the distortion-energy theory, for stress elements at $A$. (*Ans.*/ $n$ = 2.77)